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The classical relativistic two-body problem with spin and self-interactions

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Abstract. The recent classical model of a spinning Dirac particle with Zitterbewegung is generalised to two particles interacting electromagnetically. A variational principle is formulated which leads to a covariant Hamiltonian with separate centre of mass and relative terms much like the quantum two-body Dirac equation. The relative motion has the same form as the spinless case, but with the time-dependent modulated coupling constant representing the spin effects. The canonical quantisation of the theory is immediate.

1. Introduction

There has been much recent work on the classical models of single spinning particles, but very little on the interacting two-body problem to which the present work is devoted.

It is well known that classically a spinning quantum particle cannot be modelled completely as a spinning top, because the phase space of a top is much larger than the phase space of the quantum spin. The latter has therefore been modelled classically either using additional abstract non-commuting Grassmann variables [1], or using additional classical c-number spinor variables [2] to describe the dynamics of spin. In the resultant picture of the particle, a point charge performs a helical motion (called Zitterbewegung) around a fictious centre of mass which itself moves like a relativistic spinless particle. The internal orbital angular momentum of the charge around the centre of mass accounts precisely for the spin of the particle. The helical motion is the natural state of the free particle, and the charge does not radiate in this state. This picture is exactly the classical analogue of the intuitive interpretation of the Dirac electron, but now it is concrete and one can actually calculate and plot the helical trajectory. Indeed, when the classical theory is quantised, one obtains precisely the Dirac equation. The quantisation has been discussed in various forms: canonical quantisation [2] replacing Poisson brackets by commutations, Schrödinger-picture quantisation [3], path-integral quantisation [4].

The classical theory with c-number spin variables has other remarkable properties. Although the charge is pointlike, the electron acquires an affective internal structure of the size of its Compton wavelength due to the helical motion. Thus, in interaction with external electromagnetic fields which are coupled to the charge, the electron behaves as a point particle (as verified by very high energy experiments), yet it has a

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natural localisation length of the order of $1/m$. Furthermore, because of the two screw senses of the helical motion, there is the notion of antiparticles already on the classical level. Moreover, the theory can be generalised to curved spaces [5], to Kaluza-Klein spaces [6] and to spinning strings and membranes [7]. An equivalence has been established between the theories with Grassmann variables and with c-number spinors [8], the latter being more intuitive and directly related to physical variables. The purpose of this work is to generalise the one-body equations to two (or more) spinning particles interacting electromagnetically. The action principle for such a system takes into account, self-consistently, not only the mutual interactions of the particles, but also their self-interactions and radiation of the system. The latter terms are nonlinear.

This formulation again shows how close the classical theory is to the corresponding relativistic quantum theory of interacting spinning particles if the c-number spinors are employed. The classical two-body equations are, as we shall see, formally the same as the quantum equations, and go over into them after quantisation.

In section 2 we give a modified version of the one-body classical spinor equations which is better suited to the two-body problem and closer to quantum theory. Then in section 3 we derive the full two-body equations including self-energy effects, and compare the results with quantum theory.

2. Modified one-body equations

It is best to formulate the relativistic theory of a particle in terms of a covariant action principle with respect to an invariant time parameter τ. We have two sets of canonically conjugate dynamical variables, the usual pair $(x_\mu(\tau), p_\mu(\tau))$, and another pair of spin variables $(z(\tau), \bar{z}(\tau))$ which are c-number, four-component spinors. The action of the theory is (in units $c=1$)

$$A=\int \mathrm{d}\tau[\mathrm{i}\lambda\bar{z}\gamma\cdot n\dot{z}+p_\mu(\dot{x}^\mu-\bar{z}\gamma^\mu z)+eA_\mu(x)\bar{z}\gamma^\mu z]-\frac{1}{4}\int \mathrm{d}x\, F_{\mu\nu}F^{\mu\nu}. \quad (1)$$

There are two fundamental constants in the theory: λ, a constant of dimension of action ($\hbar$) and e, the charge. Mass m does not appear here and will come in later as the value of a constant of motion. The spin variables z and $\bar{z}=z^\dagger\gamma\cdot n$, with $\gamma\cdot n=\gamma^\nu n_\nu$, are thus in $\mathbb{C}^4$ (the space of 4 complex numbers). The unit timelike vector n_μ transforms like a 4-vector and indicates the choice of the time axis, and γ_μ are the Dirac matrices. In spite of the appearance of Dirac matrices, expressions like $\bar{z}\gamma^\mu z$, etc are all classical. Presently we shall pass from these complex spin variables z, $\bar{z}$ to the more physical real spin variables. The use of $(z, \bar{z})$ is very convenient for exhibiting the geometrical symplectic structure of the theory. Finally, with respect to (1), the dots are derivatives with respect to τ, x_μ are coordinates of the charge, $\dot{x}_\mu$ its velocity. The Lagrange multipliers p_μ will become the canonical momenta of the charge, which are dynamically independent from $\dot{x}_\mu$.

2.1. *Equations of motion*

The equations of motion derived from the action (1) are

$$\dot{\bar{z}}\gamma\cdot n=\frac{\mathrm{i}}{\lambda}\bar{z}\pi \qquad \gamma\cdot n\dot{z}=-\frac{\mathrm{i}}{\lambda}\pi z \qquad \dot{\pi}_\mu=eF_{\mu\nu}\dot{x}^\nu \qquad \dot{x}_\mu=\bar{z}\gamma_\mu z \quad (2)$$

for the particle variables, and

$$F^{\mu\nu}{}_{,\nu} = -e \int \bar{z}\gamma^{\mu} z \delta(x - x(\tau))\, \mathrm{d}\tau \equiv -j^{\mu} \tag{3}$$

for the electromagnetic field with $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. In (2) we have used the usual kinetic momenta

$$\pi_{\mu} \equiv p_{\mu} - eA_{\mu} \tag{4}$$

and the matrix

$$\pi \equiv \pi^{\mu}\gamma_{\mu}.$$

The system (2) is integrable. We have two constants of the motion

$$\mathcal{N} \equiv \bar{z}\gamma \cdot nz \tag{5}$$

and

$$\mathcal{H} \equiv \dot{x}_{\mu}\pi^{\mu} = \dot{x}_{\mu}(p^{\mu} - eA^{\mu}) = \bar{z}\gamma_{\mu}z(p^{\mu} - eA^{\mu}). \tag{6}$$

It is easily proved, using (2), that

$$\dot{\mathcal{N}} = 0 \qquad \text{and} \qquad \dot{\mathcal{H}} = 0 \tag{7}$$

hence the constants $\mathcal{N} = N$ and $\mathcal{H} = m$ characterise a solution.

In the previous formulation of the classical spinor model [2] the first term of the action (1) was $\mathrm{i}\lambda\bar{z}\dot{z}$ instead of $\bar{z}\gamma \cdot nz = \bar{z}\gamma^{0}z$ (for $n^{\mu} = (1000)$). This difference amounts essentially to a different choice of the invariant τ parameter. Now we have

$$\dot{x}_{0} = \frac{\mathrm{d}x^{0}}{\mathrm{d}\tau} = \bar{z}\gamma^{0}z = \text{constant} \tag{8}$$

so that we can choose τ to be the proper time parameter of the charge in the frame $n^{\mu} = (1000)$, whereas previously we had $\bar{z}z = \text{constant}$, and the corresponding time was identified with the proper time of the centre of mass. We shall see that the new form is more appropriate for many purposes, in particular in agreement with the normalisation of the Dirac wavefunction when the theory is quantised. Other features of the theory, like Zitterbewegung, remain essentially unchanged.

2.2. Exact solutions for free particles (Zitterbewegung)

We now solve the system (2) for the case $A_{\mu} = 0$. The first two equations give for the spin variables

$$\bar{z}(\tau) = \bar{z}(0) \exp(\mathrm{i}/\lambda\gamma \cdot p\gamma \cdot n\tau) = \bar{z}(0)\gamma \cdot n \exp(\mathrm{i}/\lambda\gamma \cdot n\gamma \cdot p\tau)\gamma \cdot n \tag{9}$$

$$z(\tau) = \exp(-\mathrm{i}/\lambda\gamma \cdot n\gamma \cdot p\tau)z(0) \tag{10}$$

and for the coordinates and momenta we obtain

$$p_{\mu} = \text{constant}$$

$$\dot{x}_{\mu} = \bar{z}(0)\gamma \cdot n \exp(\mathrm{i}/\lambda\gamma \cdot n\gamma \cdot p\tau)\gamma \cdot n\gamma_{\mu} \exp(-\mathrm{i}/\lambda\gamma \cdot n\gamma \cdot p\tau)z(0). \tag{11}$$

In order to exhibit the helical motion we have to evaluate the exponentials. After some algebra (see appendix 1) we obtain in units $c=1$, $\lambda=1$:

$$\begin{aligned} \bar{z}(\tau) &= \bar{z}(0)\, \mathrm{e}^{\mathrm{i}(p\cdot n)\tau}\left(\cos p_\perp\tau - \mathrm{i}\frac{p_\parallel}{p_\perp}\sin p_\perp\tau + \frac{\mathrm{i}\gamma\cdot p\gamma\cdot n}{p_\perp}\sin p_\perp\tau\right) \\ z(\tau) &= \mathrm{e}^{-\mathrm{i}(p\cdot n)\tau}\left(\cos p_\perp\tau + \mathrm{i}\frac{p_\parallel}{p_\perp}\sin p_\perp\tau - \frac{\mathrm{i}\gamma\cdot n\gamma\cdot p}{p_\perp}\sin p_\perp\tau\right)z(0) \end{aligned} \tag{12}$$

where

$$p_\perp = \sqrt{(p\cdot n)^2 - p^2} \qquad p_\parallel = (p\cdot n)$$

are the components of p_μ normal to n_μ and parallel to n_μ, respectively.

In particular, if $n^\nu = (1000)$, $p_\parallel = p_0$, $p_\perp = \sqrt{\boldsymbol{p}^2} = p$, hence

$$\bar{z}(\tau) = \bar{z}(0)\, \mathrm{e}^{\mathrm{i}p_0\tau}\left(\cos p\tau - \mathrm{i}\frac{p_0}{p}\sin p\tau + \frac{\mathrm{i}}{p}\,\gamma\cdot p\gamma^0\sin p\tau\right)$$

$$z(\tau) = \mathrm{e}^{-\mathrm{i}p_0\tau}\left(\cos p\tau + \mathrm{i}\frac{p_0}{p}\sin p\tau - \frac{\mathrm{i}}{p}\,\gamma^0\gamma\cdot p\sin p\tau\right)z(0).$$

In the rest frame of the particle $\boldsymbol{p}=\boldsymbol{0}$, z and $\bar{z}$ have simple oscillations

$$\bar{z}(\tau) = \bar{z}(0)\,\mathrm{e}^{\mathrm{i}m\tau} \qquad z(\tau) = \mathrm{e}^{-\mathrm{i}m\tau}z(0).$$

The helical orbit itself is given by its velocity vector $\dot{x}_\mu = \bar{z}\gamma_\mu z$ which we evaluate using (12). After some lengthy calculation the result is (cf appendix 2)

$$\dot{x}_\mu(\tau) = \dot{x}_\mu(0) + \ddot{x}_\mu(0)\frac{\sin 2p_\perp\tau}{2p_\perp} + 2\frac{\sin^2 p_\perp\tau}{p_\perp^2}V_\mu \tag{13}$$

which can easily be integrated again. Here the constant vector V_μ is

$$V_\mu = n_\mu(\mathscr{H}p_\parallel - p^2\mathscr{N}) - p_\mu(\mathscr{H} - p_\parallel\mathscr{N}).$$

Thus, the orbit is given by the initial position, initial velocity and initial acceleration for a given particle with $\mathscr{H}=m$, $\mathscr{N}=N$ (which we can take equal to 1) and momentum p^μ.

2.3. Symplectic structure

Our dynamical system is symplectic with the Poisson brackets

$$\{f,g\} = \mathrm{i}\left(\frac{\partial f}{\partial\bar{z}}\frac{\partial g}{\partial z}\gamma\cdot n - \gamma\cdot n\frac{\partial f}{\partial z}\frac{\partial g}{\partial\bar{z}}\right) + g^{\mu\nu}\left(\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial p^\nu} - \frac{\partial f}{\partial p^\mu}\frac{\partial g}{\partial x^\nu}\right). \tag{14}$$

Indeed, both commutator and Hamilton's equations are satisfied with our Hamiltonian $\mathscr{H} = \bar{z}\gamma^\mu z\pi_\mu$, with $\pi_\mu = p_\mu - eA_\mu$:

$$\begin{aligned} \dot{\bar{z}} &= \{\bar{z},\mathscr{H}\} = \mathrm{i}\frac{\partial\mathscr{H}}{\partial z}\gamma\cdot n = \mathrm{i}\bar{z}\pi\gamma\cdot n \\ \dot{z} &= \{z,\mathscr{H}\} = -\mathrm{i}\gamma\cdot n\frac{\partial\mathscr{H}}{\partial\bar{z}} = -\mathrm{i}\gamma\cdot n\pi z \\ \dot{\pi}_\mu &= \{\pi_\mu,\mathscr{H}\} = -eA_{\mu,\nu}\frac{\partial\mathscr{H}}{\partial p^\nu} - \frac{\partial\mathscr{H}}{\partial x^\mu} = eF_{\mu\nu}\dot{x}^\nu \end{aligned} \tag{15}$$

$$\dot{x}_\mu = \{x_\mu, \mathcal{H}\} = \frac{\partial \mathcal{H}}{\partial p^\mu} = \bar{z}\gamma^\mu z$$

which are the equations (2).

2.4. Spin variables

We can use as new dynamical variables, instead of z, $\bar{z}$ the real and imaginary parts of z. Or, as in previous work [2], we can use the variables $(x_\mu, p_\mu, v_\mu, S_{\mu\nu})$, where

$$v_\mu = \dot{x}_\mu = \bar{z}\gamma_\mu z \tag{16}$$

and the spin tensor $S_{\mu\nu}$ is defined by

$$S_{\mu\nu} = \frac{\mathrm{i}}{4}\bar{z}[\gamma_\mu, \gamma_\nu]^* z. \tag{17}$$

The deformed commutator $[\ ,\ ]^*$ is defined by

$$[\gamma_\mu, \gamma_\nu]^* = \gamma_\mu \gamma \cdot n \gamma_\nu - \gamma_\nu \gamma \cdot n \gamma_\mu .$$

We have then

$$\begin{aligned} \dot{v} &= \dot{\bar{z}}\gamma_\mu z + \bar{z}\gamma_\mu \dot{z} = \mathrm{i}\bar{z}\gamma^\nu \pi_\nu \gamma \cdot n \gamma_\mu z - \mathrm{i}\bar{z}\gamma_\mu \gamma \cdot n \gamma_\nu \pi^\nu z \\ &= -4S_{\mu\nu}\pi^\nu \end{aligned} \tag{18}$$

and

$$S_{\mu\nu} = -\tfrac{1}{4}\bar{z}[\gamma_\alpha, \gamma_\nu]^*]^* z \pi^\alpha \tag{19}$$

which can also be written as

$$S^{\mu\nu} = \bar{z}\gamma \cdot n(p_\mu \gamma_\nu - p_\nu \gamma_\mu)z + (p \cdot n)\bar{z}\gamma \cdot n(n_\mu \gamma_\nu - n_\nu \gamma_\mu)z + (p_\mu n_\nu - p_\nu n_\mu)\bar{z}z. \tag{19'}$$

2.5. One-body self-energy term

Equation (2) shows that π_μ and $\dot{x}_\mu$ are independent dynamical variables. In the presence of an external field, π_μ is no longer constant, so we must solve the four coupled equations in (2). Choosing a gauge A^μ, $\mu = 0$, equation (3) becomes

$$\Box A_\mu(x) = j_\mu = e \int \mathrm{d}\tau\, \bar{z}\gamma_\mu z \delta(x - x(\tau))$$

which can be solved as

$$\begin{aligned} A_\mu(x) &= e \int \mathrm{d}y\, D(x-y)\bar{z}\gamma_\mu z \delta(y - x(\tau))\, \mathrm{d}\tau + A_\mu^{\text{ext}} \\ &= e \int \mathrm{d}\tau\, D(x - x(\tau))\bar{z}\gamma^\mu z + A_\mu^{\text{ext}} \end{aligned} \tag{20}$$

where $D(x-y)$ is the Green function of the d'Alembertian $\Box$. If we insert (20) into the action (1), at the same time replacing the last term in (1), and by partial integration as

$$-\frac{1}{4}\int \mathrm{d}x\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\int \mathrm{d}x\, A_\mu j^\mu$$

we can rewrite our action (1) in the form

$$A=\int \mathrm{d}\tau(\mathrm{i}\lambda\bar{z}\gamma\cdot n\dot{z}+p_\mu\dot{x}^\mu-p_\mu\bar{z}\gamma^\mu z+eA^{\mathrm{ext}}_\mu\bar{z}\gamma^\mu z)$$
$$-\frac{e^2}{2}\int \mathrm{d}\tau\,\mathrm{d}\tau'\,\bar{z}(\tau)\gamma^\mu z(\tau)D(x(\tau)-x(\tau'))\bar{z}(\tau')\gamma_\mu z(\tau'). \tag{21}$$

From here it is possible to derive the generalisation of the Lienard-Wiechert potentials and the Lorentz-Dirac equation to spinning particles. This has been done recently [9]. The result is that the classical Lorentz-Dirac equation

$$m\ddot{x}_\mu=eF^{\mathrm{ext}}_{\mu\nu}\dot{x}^\nu+\tfrac{2}{3}e^2(\dddot{x}_\mu+(\ddot{x})^2\dot{x}_\mu)$$
$$=eF^{\mathrm{ext}}_{\mu\nu}v^\nu+\tfrac{2}{3}e^2\tilde{g}_{\mu\nu}\ddot{v}^\nu \qquad \tilde{g}_{\mu\nu}=g_{\mu\nu}-\frac{v_\mu v_\nu}{v^2} \qquad v_\mu\equiv\dot{x}_\mu \tag{22}$$

acquires a new spin term and goes over to

$$\dot{\pi}_\mu=eF^{\mathrm{ext}}_{\mu\nu}v^\nu+e^2\tilde{g}_{\mu\nu}\left(\frac{2}{3}\frac{\ddot{v}^\nu}{v^2}-\frac{9}{4}\frac{(v\cdot\dot{v})\dot{v}^\nu}{v^4}\right). \tag{23}$$

In the limit $v^2\to 1$, $v\cdot\dot{v}\to 0$, $\dot{\pi}_\mu\to m\ddot{x}_\mu$ we recover the original equation (22).

3. Relativistic two-body system with self-interaction

We now start, instead of (1), from the action for two particles (the generalisation to many particles is immediate)

$$A=\int \mathrm{d}\tau_1[\mathrm{i}\lambda_1\bar{z}_1\gamma\cdot n\dot{z}_1+p_{1\mu}(\dot{x}^\mu_1-\bar{z}_1\gamma^\mu z_1)]+\int \mathrm{d}\tau_2[\mathrm{i}\lambda_2\bar{z}_2\gamma\cdot n\dot{z}_2+p_{2\mu}(\dot{x}^\mu_2-\bar{z}_2\gamma^\mu z_2)]$$
$$+\int \mathrm{d}x\bigg[\bigg(\int \mathrm{d}\tau_1\,e_1\bar{z}_1\gamma^\mu z_1\cdot\delta(x-x_1)$$
$$+\int \mathrm{d}\tau_2\,e_2\bar{z}_2\gamma^\mu z_2\delta(x-x_2)\bigg)A_\mu(x)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\bigg]. \tag{24}$$

The field equations

$$F^{\mu\nu}{}_{,\nu}=-j_\mu=-\int e_1\bar{z}_1\gamma^\mu z_1\delta(x-x_1)\,\mathrm{d}\tau_1-\int e_2\bar{z}_2\gamma^\mu z_2\delta(x-x_2)\,\mathrm{d}\tau_2$$

can be solved in the gauge $A^\mu{}_{,\mu}=0$ to give

$$A_\mu(x)=\int \mathrm{d}y\,D(x-y)\left(\int e_1\bar{z}_1\gamma^\mu z_1\delta(y-x_1)\,\mathrm{d}\tau_1+\int e_2\bar{z}_2\gamma^\mu z_2\delta(y-x_2)\,\mathrm{d}\tau_2\right)$$
$$=\int e_1\bar{z}_1\gamma^\mu z_1D(x-x_1(\tau_1))\,\mathrm{d}\tau_1+\int e_2\bar{z}_2\gamma^\mu z_2D(x-x_2(\tau_2))\,\mathrm{d}\tau_2. \tag{25}$$

We insert this into (21). In the last term we make an integration by parts

$$-\frac{1}{4}\int \mathrm{d}x\,F_{\mu\nu}F^{\mu\nu}=\frac{1}{2}\int \mathrm{d}x\,j_\mu A^\mu.$$

We then obtain the 'action at a distance'

$$A=\int \mathrm{d}\tau_1[\mathrm{i}\lambda_1\bar{z}_1\gamma\cdot n\dot{z}_1+p_{1\mu}(\dot{x}_1^\mu-\bar{z}_1\gamma^\mu z_1)]+\int \mathrm{d}\tau_2[\mathrm{i}\lambda_2\bar{z}_2\gamma\cdot n\dot{z}_2+p_{2\mu}(\dot{x}_2^\mu-\bar{z}_2\gamma^\mu z_2)]$$
$$-e_1e_2\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\bar{z}_1\gamma^\mu z_1 D(x_1-x_2)\bar{z}_2\gamma_\mu z_2$$
$$-\frac{e_1^2}{2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_1'\,\bar{z}_1\gamma^\mu z_1 D(x_1(\tau_1)-x_1(\tau_1'))\bar{z}_1\gamma_\mu z_1$$
$$-\frac{e_2^2}{2}\int \mathrm{d}\tau_2\,\mathrm{d}\tau_2'\,\bar{z}_2\gamma^\mu z_2 D(x_2(\tau_2)-x_2(\tau_2'))\bar{z}_2\gamma_\mu z_2. \qquad (26)$$

The first two terms are the kinetic energies of the particles, the third term is the mutual interaction of two particles, and the last two terms the self-energies of each particle. The arguments of z_1 and z_2 are clear from the integrals. The spin spaces of each particle should be distinguished: $\gamma^{\mu(1)}$ and $\gamma^{\mu(2)}$, but the superscripts (1) and (2) are clear from the context, so we omit them.

Variation of the action with respect to particle coordinates 1 and 2 would give complicated coupled equations. Instead we shall discuss a variational principle for the two-body system as a whole in analogy to the quantum theory [10].

3.1. Centre-of-mass and relative coordinates

For this purpose we introduce composite spinors

$$Z(\tau_1,\tau_2)=z_1(\tau_1)\otimes z_2(\tau_2) \qquad \bar{Z}(\tau_1,\tau_2)=\bar{z}_1(\tau_1)\otimes\bar{z}_2(\tau_2). \qquad (27)$$

These are 16-component spinors, direct products of the two spinor spaces. We shall now rewrite the action (24) in terms of these composite spinors. This is straightforward for the third term. But in order to transform the remaining terms we normalise our spinors such that

$$\int \mathrm{d}\tau_1\,\bar{z}_1\gamma\cdot nz_1\delta(\tau_1-\tau)=1 \qquad (28)$$

for every τ, or simply $N_i=\bar{z}_i\gamma\cdot nz_i=1$ $(i=1,2)$. We now multiply each kinetic energy term in (1) by the normalisation integral of the other particle at equal times. Similarly in the self-energy terms we multiply each term *twice* by the normalisation integrals of the other particle. Then the action can be brought to the following form:

$$A=\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\bigg[\mathrm{i}\bar{Z}\gamma\cdot n\otimes\gamma\cdot n\left(\frac{\partial Z}{\partial\tau_1}+\frac{\partial Z}{\partial\tau_2}\right)+p_1\cdot\dot{X}_1+p_2\cdot\dot{X}_2$$
$$-\bar{Z}(\gamma\cdot p_1\otimes\gamma\cdot n+\gamma\cdot n\otimes\gamma\cdot p_2)Z\bigg]$$
$$\times\delta(\tau_1-\tau_2)-e_1e_2\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\bar{Z}\gamma_\mu\otimes\gamma^\mu ZD(x_1-x_2)$$
$$-\frac{1}{2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1'\,\mathrm{d}\tau_2'[e_1^2\bar{Z}\gamma_\mu\otimes\gamma\cdot nZD(x_1-x_1(\tau_1'))$$
$$\times\bar{Z}(\tau_1',\tau_2')\gamma^\mu\otimes\gamma\cdot nZ(\tau_1',\tau_2')$$
$$+e_2^2\bar{Z}\gamma\cdot n\otimes\gamma^\mu ZD(x_2-x_2(\tau_2')\bar{Z}(\tau_1'\tau_2')\gamma\cdot n\otimes\gamma^\mu Z(\tau_1'\tau_2')]$$
$$\times\delta(\tau_1-\tau_2)\delta(\tau_1'-\tau_2') \qquad (29)$$

where $\dot{X}_1 = \dot{x}_1 \bar{z}_2 \gamma \cdot n z_2$ and $\dot{X}_2 = \dot{x}_2 \bar{z}_1 \gamma \cdot n z_1$. The spin matrices are always written as a tensor product, the first factor referring to particle 1, the second to particle 2.

This action looks like a two-times action, but we shall now show that it actually reduces to a one-time action, and hence to a one-time equation for both particles, as in the quantum case [10].

We now introduce the standard centre-of-mass and relative coordinates and momenta by

$$x_{1\mu} = X_\mu + (1-a)x_\mu \qquad x_{2\mu} = X_\mu - a x_\mu \qquad a = m_1/(m_1+m_2) \tag{30}$$

and

$$p_{1\mu} = aP_\mu + p_\mu \qquad p_{2\mu} = (1-a)P_\mu - p_\mu$$

together with the transformation of the invariant time parameters

$$\tau_1 = T + (1-a)\tau \qquad \tau_2 = T - a\tau \qquad \text{with} \qquad \mathrm{d}\tau_1\,\mathrm{d}\tau_2 = \mathrm{d}T\,\mathrm{d}\tau. \tag{31}$$

The composite spinor $Z(\tau_1, \tau_2)$ will be a function $Z(T, \tau)$ in the new variables. We obtain then by the replacements

$$\begin{aligned} &\frac{\partial}{\partial \tau_1} = \frac{\partial}{\partial \tau} + a\frac{\partial}{\partial T} \qquad \frac{\partial}{\partial \tau_2} = -\frac{\partial}{\partial \tau} + (1-a)\frac{\partial}{\partial T} \\ &\frac{\partial Z}{\partial \tau_1} + \frac{\partial Z}{\partial \tau_2} = \frac{\partial Z}{\partial T}. \end{aligned} \tag{32}$$

Furthermore, because each particle has its own time τ_1 and τ_2, we have $\mathrm{d}x_1/\mathrm{d}\tau_2 = 0$, and $\mathrm{d}x_2/\mathrm{d}\tau_1 = 0$; hence we find

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau_1} = \frac{\mathrm{d}x_1}{\mathrm{d}T} \qquad \frac{\mathrm{d}x_2}{\mathrm{d}\tau_2} = \frac{\mathrm{d}x_2}{\mathrm{d}T}. \tag{33}$$

Thus all relevant derivatives are with respect to T; hence we can integrate over relative times τ and τ'.

In the second integral, we can decompose covariantly with respect to the normal n^μ to a spacelike surface

$$(x_1 - x_2)^2 = x^2 = (x \cdot n)^2 - x_\perp^2.$$

Hence

$$\delta(x^2) = \frac{\delta((x\cdot n) - x_\perp) + \delta((x \cdot n) + x_\perp)}{x_\perp} \tag{34}$$

where $x_\perp = [(x\cdot n)^2 - x^2]^{1/2}$ is the relativistic relative distance.

With these steps, our action depends only on the COM time T

$$\begin{aligned} A = \int \mathrm{d}T\Big(&\mathrm{i}\bar{Z}\gamma\cdot n \otimes \gamma \cdot n\dot{Z} + P_\mu(\dot{X}_\mu - \bar{Z}\Gamma^\mu Z) + p_\mu(\dot{x}^\mu - \bar{Z}k^\mu Z) - \frac{e_1 e_2}{x_\perp}\bar{Z}\gamma^\mu \otimes \gamma_\mu Z\Big) \\ &-\frac{1}{2}\int \mathrm{d}T\,\mathrm{d}T'[e_1^2 \bar{Z}\gamma_\mu \otimes \gamma\cdot nZD\{X - X(T') + (1-a)(x - x(T')\} \\ &\times \bar{Z}(T')\gamma^\mu \otimes \gamma \cdot nZ(T') \\ &+ e_2^2\bar{Z}\gamma\cdot n \otimes \gamma^\mu Z D\{X - X(T') - a(x - x(T')\}\bar{Z}(T')\gamma\cdot n_0 \gamma^\mu Z(T')] \end{aligned} \tag{35}$$

where

$$\Gamma^\mu = a\gamma^\mu \otimes \gamma\cdot n + (1-a)\gamma\cdot n \otimes \gamma^\mu$$
$$k^\mu = \gamma^\mu \otimes \gamma\cdot n - \gamma\cdot n\otimes\gamma^\mu. \tag{36}$$

The fact that the relative time τ drops out can also be seen from the relation $k^\mu \cdot n_\mu = 0$. Hence only the perpendicular component of the relative momentum p_μ to the spacelike surface appears for the equation

$$\bar{Z}p_\mu k^\mu Z = \bar{Z}p^\mu_\perp k_{\mu\perp} Z.$$

We can rewrite the action as

$$A = \int \mathrm{d}T[\mathrm{i}\bar{Z}\gamma\cdot n\otimes\gamma\cdot n\dot{Z} + P_\mu \dot{X}^\mu + p_\mu \dot{x}^\mu - \mathscr{H}] \tag{37}$$

where the covariant Hamiltonian $\mathscr{H}$ is given by

$$\mathscr{H} = \bar{Z}\left(\Gamma^\mu P_\mu + k^\mu_\perp p_{\mu\perp} + \frac{e_1 e_2}{x_\perp}\gamma^\mu\otimes\gamma_\mu + V_{\text{self}}\right)Z$$
$$\equiv \bar{Z}HZ. \tag{38}$$

Note that $p\cdot\dot{x} = (p\cdot n)(\dot{x}\cdot n) + p_\perp\cdot\dot{x}_\perp = p_\perp\cdot\dot{x}_\perp$. Hence, the dynamical variables are P_μ, X_μ for the centre-of-mass motion and $P_{\mu\perp}$, $X_{\mu\perp}$ for the relative motion, and the self potential is given by

$$V^{\text{self}} = \tfrac{1}{2}e_1\gamma_\mu\otimes\gamma\cdot n\int \mathrm{d}T'\, D\{X - X(T') + (1-a)(x - x(T'))\}\bar{Z}(T')\gamma^\mu\otimes\gamma\cdot nZ(T')$$
$$+\tfrac{1}{2}e_2\gamma\cdot n\otimes\gamma_\mu\int \mathrm{d}T'\, D\{X - X(T') - a(x-x)(T'))\}$$
$$\times\bar{Z}(T')\gamma\cdot n\otimes\gamma_\mu Z(T'). \tag{39}$$

The two-body equations of motions are now

$$\dot{X}^\mu = \frac{\partial\mathscr{H}}{\partial p_\mu} = \bar{Z}\Gamma^\mu Z$$
$$\dot{x}^\mu = \frac{\partial\mathscr{H}}{\partial p_\mu} = \bar{Z}k^\mu Z \qquad \text{with } \dot{x}\cdot n = 0$$
$$\dot{P}_\mu = \frac{\partial\mathscr{H}}{\partial X^\mu} = \bar{Z}\frac{\partial V^{\text{self}}}{\partial X^\mu}Z \tag{40}$$
$$\dot{p}_{\mu\perp} = -\frac{\partial\mathscr{H}}{\partial\dot{x}^\mu} = e_1 e_2\bar{Z}\gamma^\mu\otimes\gamma_\mu Z_1\frac{x_{\perp\mu}}{x^3_\perp} + \bar{Z}\frac{\partial V^{\text{self}}}{\partial x^\mu}Z$$
$$\mathrm{i}\dot{\bar{Z}}\gamma\cdot n\otimes\gamma\cdot n = -\frac{\partial\mathscr{H}}{\partial Z} = -\bar{Z}H$$
$$\mathrm{i}\gamma\cdot n\otimes\gamma\cdot n\dot{Z} = \frac{\partial\mathscr{H}}{\partial\bar{Z}} = HZ.$$

The constants of the motion are, as in one-body problem,

$$\mathscr{N} = \bar{Z}\gamma\cdot n\otimes\gamma\cdot nZ \qquad \mathscr{H} = \bar{Z}HZ.$$

We define the value of $\mathscr{H}$ to be the total rest mass of the system

$$\mathscr{H} = M \tag{41}$$

and the value of $\mathcal{N}$ can be taken to be unity by the normalisation of spinors. If we first disregard the (small) radiative effects in V^{self}, then (40) shows that the total momentum P_μ is constant, and the relative motion is transversal to the time vector n_μ and is given by

$$\dot{p}_{\mu\perp} = (\bar{Z}\gamma^\mu \otimes \gamma_\mu Z) e_1 e_2 \frac{x_{\mu\perp}}{x_\perp^3}. \tag{42}$$

For $n_\mu = (1000)$, this is simply

$$\dot{\boldsymbol{p}} = (\bar{Z}\gamma^\mu \otimes \gamma_\mu Z) e_1 e_2 \frac{\boldsymbol{r}}{r^3}. \tag{42'}$$

We thus obtain the important result that the effect of spin on the relative motion is to multiply the coupling constant $e_1 e_2$ by the time-dependent (oscillatory) spin factor $(\bar{Z}\gamma^\mu \otimes \gamma_\mu Z)$. As a result the orbit will be modified by the Zitterbewegung.

In order to write the equations in ordinary time t, instead of the invariant time τ, we start from (41), neglecting V^{self}, i.e.

$$\mathscr{H} = \bar{Z}\Gamma^\mu Z P_\mu + \bar{Z} k_\perp^\mu Z p_{\mu\perp} + \bar{Z}\gamma^\mu \otimes \gamma_\mu Z \frac{e_1 e_2}{x_\perp} = M. \tag{43}$$

This is a classical Hamiltonian, but linear in both the total momentum P_μ and the relative momentum $p_{\mu\perp}$. The Hamiltonian in ordinary time can be identified with $P_\parallel$, i.e. P_0 if $n_\mu = (1000)$. Since $\bar{Z}\Gamma^\parallel Z = \bar{Z}\gamma\cdot n \otimes \gamma \cdot n Z = \mathcal{N}$ we obtain

$$(P\cdot n) \equiv P_\parallel = \left(\bar{Z}\Gamma_\perp^\mu Z P_{\mu\perp} - \bar{Z} k_\perp^\mu Z p_{\mu\perp} - \bar{Z}\gamma^\mu \otimes \gamma_\mu Z \left| \frac{e_1 e_2}{\Gamma_\perp} + M \right) \mathcal{N}^{-1} \tag{44}$$

or, for $n_\mu = (1000)$ and choosing the constant of motion $\mathcal{N} = 1$,

$$P_0 = H_0 = \bar{Z}\boldsymbol{\Gamma} Z \cdot \boldsymbol{P} + \bar{Z}(\boldsymbol{\gamma}_1 \cdot \gamma_0 - \gamma_0 \cdot \boldsymbol{\gamma}_2) Z \cdot \boldsymbol{p} - e_1 e_2 \bar{Z}(\gamma_1^\mu \otimes \gamma_\mu^2) Z \cdot \frac{1}{r} + M. \tag{44'}$$

This is exactly of the same form as the two-body Dirac Hamiltonian operator [10]

$$H_{\text{D}} = \boldsymbol{\Gamma}\cdot\boldsymbol{P} + (\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2)\cdot\boldsymbol{p} - e_1 e_2 \frac{1 - \boldsymbol{\alpha}_1 \cdot \boldsymbol{\alpha}_2}{r} + m_1\beta_1 + m_2\beta_2 \tag{45}$$

but now, in the classical case, the coefficient of $\boldsymbol{P}$, $\boldsymbol{p}$ and $1/r$ are c-numbers, but time-dependent quantities, instead of 16×16 matrices.

Equation (40) or the Hamiltonian (43) can easily be quantised as in the one-particle case. One has only to replace the Poisson brackets by commutators.

3.2. Limit to one-body spinning particles

If particle 2 is very heavy, i.e.

$$m_2 \to \infty \qquad a = \frac{m_1}{m_1 + m_2} \to 0$$

we have the limits

$$\dot{x}_2^0 = \bar{z}_2 \gamma^0 z_2 \to 1 \qquad \dot{\boldsymbol{x}}_2 = \bar{z}_2 \boldsymbol{\gamma} z_2 \to 0$$

and

$$p_2^0 = m_2 \qquad \mathbf{p}_2 = 0$$

and if we also neglect the spin of the heavy particle at rest, i.e.

$$\bar{z}_2\gamma \cdot n\dot{z}_2 \to 0$$

then the two-body action (26) goes over to ($\lambda_1 = 1$, $c = 1$)

$$A = \int \mathrm{d}\tau[\mathrm{i}\bar{z}_1\gamma \cdot n\dot{z}_1 + p_1(\dot{x}_1^\mu - \bar{z}_1\gamma^\mu z_1)] - e_1e_2 \int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\bar{z}_1\gamma^0 z_1 D(x_1(\tau_1) - x_2(\tau_2))$$

$$-\frac{e_1^2}{2}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_1^1\,\bar{z}_1\gamma^\mu z_1 D(x_1(\tau_1) - x_1(\tau_1^1))\bar{z}_1\gamma_\mu z_1$$

which is the action (21) for particle 1 in the field $A_0^{\mathrm{ext}}(x_1) = e_2 \int \mathrm{d}\tau_2\, D(x_1 - x_2(\tau_2))$ of the second particle at rest.

3.3. *Limit to spinless particles*

If both particles are spinless we replace

$$\bar{z}_i\gamma \cdot n\dot{z}_i \to 0 \qquad i = 1, 2$$

i.e. no spin kinetic energy, and

$$v_\mu^i = \bar{z}_i\gamma_\mu z_i \to \frac{1}{m_i}(p_{i\mu} - eA_\mu(x_i))\bar{z}_i\gamma \cdot nz_i$$

$$= \frac{\lambda_i}{m_i}(p_{i\mu} - eA_\mu(x_i))$$

where λ_i are the normalisation constants of the spinors. Then the action (24) goes over into

$$A = \int \mathrm{d}\tau_i\left(p_i \cdot \dot{x}_i - p_i \cdot \frac{\lambda_i}{m_i}(p_i - e_iA(x_i)) + \frac{\lambda_i e_i}{m}(p_i - e_iA) \cdot A(x_i)\right) - \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}x$$

or

$$A = \int \mathrm{d}\tau_i\left(p_i \cdot \dot{x}_i - \frac{\lambda_i}{m_i}p_i^2 + \frac{2e\lambda_i}{m_i}p_i \cdot A - \frac{\lambda_i e_i^2}{m_i}A^2\right) - \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}x.$$

This is precisely the phase-space action of spinless particles [4], which is equivalent to the configuration-space action

$$A = \int \mathrm{d}\tau_i(\tfrac{1}{2}m_i\dot{x}_i^2 + e_i\dot{x}_i \cdot A(x_i)) - \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}x$$

if we use $\lambda_i = \frac{1}{2}$ and $\dot{x}_\mu = (1/m)(p_\mu - eA_\mu)$, or $p_\mu = m\dot{x}_\mu + eA_\mu$.

Appendix 1

We wish to expand the exponential $\exp(\mathrm{i}\gamma \cdot p\gamma \cdot ns)$. With $\gamma \cdot p = A$, $\gamma \cdot n = B$, we have

$$\exp(\mathrm{i}\gamma \cdot p\gamma \cdot ns) = \exp(\mathrm{i}ABs) = 1 + \frac{\mathrm{i}s}{1!}AB + \frac{(is)^2}{2!}ABAB + \ldots.$$

Setting $A^2=a^2$ and $B^2=b^2$ and $AB+BA=2(p\cdot n)=c$, we find

$$(AB)^2=cAB-a^2b^2$$
$$(AB)^3=(c^2-a^2b^2)AB-ca^2b^2$$
$$\vdots$$
$$(AB)^n=f_nAB+g_n$$
$$(AB)^{n+1}=(f_nc+g_n)AB-f_na^2b^2=f_{n+1}AB+g_{n+1}=(AB\ 1)\begin{pmatrix}f_{n+1}\\ g_{n+1}\end{pmatrix}$$

so that we have the recursion formula

$$\begin{pmatrix}f_{n+1}\\ g_{n+1}\end{pmatrix}=\begin{pmatrix}c & 1\\ -a^2b^2 & 0\end{pmatrix}\begin{pmatrix}f_n\\ g_n\end{pmatrix}\equiv M\begin{pmatrix}f_n\\ g_n\end{pmatrix}.$$

Thus with

$$\begin{pmatrix}f_0\\ g_0\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$$

we can write

$$\exp(\mathrm{i}ABs)=(AB\ 1)\ \mathrm{e}i^{Ms}\begin{pmatrix}0\\ 1\end{pmatrix}.$$

Since M is now a numerical 2×2 matrix, in our case,

$$M=(p\cdot n)+\boldsymbol{\xi}\cdot\boldsymbol{\sigma}$$
$$\xi_1=\tfrac{1}{2}(1-p^2)\qquad \xi_2=\tfrac{1}{2}\mathrm{i}(1+p^2)\qquad \xi_3=(p\cdot n)$$
$$|\boldsymbol{\xi}|^2=(p\cdot n)^2-p^2=p_\perp^2$$

we obtain

$$\mathrm{e}^{\mathrm{i}Ms}=\mathrm{e}^{\mathrm{i}(p\cdot n)s}\left(\cos p_\perp s+\mathrm{i}\frac{\boldsymbol{\xi}\cdot\boldsymbol{\sigma}}{p_\perp}\sin p_\perp s\right).$$

Consequently

$$\exp(\mathrm{i}ABs)=(AB\ 1)\ \mathrm{e}^{\mathrm{i}(p\cdot n)s}\left(\cos p_\perp s+\frac{\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{\sigma}}{p_\perp}\sin p_\perp s\right)\begin{pmatrix}0\\ 1\end{pmatrix}$$
$$=\mathrm{e}^{\mathrm{i}(p\cdot n)s}\left(\cos p_\perp s-\mathrm{i}\frac{(p\cdot n)}{p_\perp}\sin p_\perp s+\mathrm{i}\frac{\gamma\cdot p\gamma\cdot n}{p_\perp}\sin p_\perp s\right).$$

Appendix 2

For any quantity $\bar{z}\mathscr{A}z$ we obtain its time development, using (12)

$$\mathscr{A}(\tau)=\bar{z}(\tau)\mathscr{A}z(\tau)$$
$$=\bar{z}(0)\left(\cos p_\perp\tau-\mathrm{i}\frac{p_\parallel}{p_\perp}\sin p_\perp\tau+\frac{\mathrm{i}}{p_\perp}\gamma\cdot p\gamma\cdot n\sin p_\perp\tau\right)\mathscr{A}$$
$$\times\left(\cos p_\perp\tau+\mathrm{i}\frac{p_\parallel}{p_\perp}\sin p_\perp\tau-\frac{\mathrm{i}}{p_\perp}\gamma\cdot n\gamma\cdot p\sin p\perp\tau\right)z(0).$$

This expression may be evaluated to give

$$\mathscr{A}(\tau)=\mathscr{A}(0)+\dot{\mathscr{A}}(0)\frac{\sin 2p_\perp\tau}{2p_\perp}+\mathscr{A}(0)\frac{p_\parallel^2-p_\perp^2}{p_\perp^2}\sin^2 p_\perp\tau$$
$$-\bar{z}(0)\left(\frac{p_\parallel}{p_\perp^2}(\mathscr{A}\gamma\cdot n\gamma\cdot p+\gamma\cdot p\gamma\cdot n\mathscr{A})\sin^2 p_\perp\tau\right.$$
$$\left.-\frac{1}{p_\perp^2}\gamma\cdot p\gamma\cdot n\mathscr{A}\gamma\cdot n\gamma\cdot p\sin^2 p_\perp\tau\right)z(0).$$

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